IMPROVING THE BEST KNOWN CONSTANTS FOR THE REAL MULTILINEAR BOHNENBLUST–HILLE INEQUALITY: AN EXHAUSTIVE COMBINATORIAL APPROACH

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Abstract. Since the proof of the multilinear Bohnenblust–Hille inequality (*Annals of Mathematics*, 1931), the constants involved have been improved along the decades. It was recently shown that these constants can be chosen of the form $\sqrt{2}\,(n-1)^{0.526322}$ for real scalars and $\frac{2}{\sqrt{\pi}}\,(n-1)^{0.304975}$ for complex scalars. However, these formulas are just approximations of complicated recursive formulas, which are, up to now, the best known estimates for the multilinear Bohnenblust–Hille constants. As a consequence of the results of this paper we show that, for real scalars the best known constants are not optimal.

1. Introduction

Let $\mathbb{K}$ be the real or complex scalar field. The multilinear Bohnenblust–Hille inequality ([1]) asserts that for every positive integer $n \geq 1$ there exists a sequence of positive scalars $(B_n)_{n=1}^{\infty}$ in $[1,\infty)$ such that

$$\left(\sum_{i_1,\ldots,i_n=1}^{N}\left|U(e_{i_1},\ldots,e_{i_n})\right|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}} \leq B_n \sup_{z_1,\ldots,z_n\in\mathbb{D}^N}\left|U(z_1,\ldots,z_n)\right| \tag{1.1}$$

for all n-linear forms $U:\mathbb{K}^N\times\cdots\times\mathbb{K}^N\to\mathbb{K}$ and every positive integer N, where $(e_i)_{i=1}^{N}$ denotes the canonical basis of $\mathbb{K}^N$ and $\mathbb{D}^N$ represents the open unit polydisc in $\mathbb{K}^N$. The exact values for the optimal constants B_n satisfying (1.1) remains a mystery and are being improved along the time. The first estimates ([1, 2, 7, 14]) suggested an exponential growth and only very recently quite different results have arisen. The ultimate information related to the search of optimal values for constants satisfying (1.1) is:

- For real scalars,

  $$2^{1-\frac{1}{n}} \leq B_n \leq C_n,$$

  where $(C_n)_{n=1}^{\infty}$ is given by a puzzling recursive formula (see (2.3)). Some approximations of the values of C_n allows us to conclude that

  $$2^{1-\frac{1}{n}} \leq B_n \leq \sqrt{2}\,(n-1)^{\log_2\left(\frac{e^{1-\frac{\gamma}{2}}}{\sqrt{2}}\right)} \leq \sqrt{2}\,(n-1)^{0.526322}.$$

  The above results are taken from [5] (lower estimates) and [11, 12] (upper estimates). We also refer to [15] for a different approach and to [8] for applications in Quantum Information Theory.
- For complex scalars,

$$1 \leq B_n \leq \widetilde{C}_n, \tag{1.2}$$

*Supported by Capes.
**Supported by CNPq Edital Universal and INCT-Matemtica.
***Supported by grant MTM2012-34341.

where $\left(\widetilde{C}_n\right)_{n=1}^{\infty}$ is given by a similar recursive formula (2.4). Some approximations of the values of C_n shows that

$$1 \leq B_n \leq \frac{2}{\sqrt{\pi}}\,(n-1)^{\log_2\left(e^{\frac{1}{2}-\frac{1}{2}\gamma}\right)} \leq \frac{2}{\sqrt{\pi}}\,(n-1)^{0.304975}. \tag{1.3}$$

These results can be found in [10, 12]. Above, the letter γ denotes Euler–Mascheroni's constant, which is approximately 0.5772.

Denoting by K_n the optimal constants satisfying the (real) multilinear Bohnenblust–Hille inequality it is still an open problem if $K_n = 2^{1-\frac{1}{n}}$ or $K_n = C_n$ or whether K_n lies strictly between these bounds. The only known precise value appears in the case $n = 2$, since $2^{1-\frac{1}{2}} = C_2$. For the complex case, the similar question is unsolved for the estimates (1.2).

The main goal of this note is to show that the best known upper estimates $(C_n)_{n=1}^{\infty}$ for the real multilinear Bohnenblust–Hille inequality are not optimal. The case of complex scalars is also investigated by the same technique but our results were not conclusive.

2. The best known formulas

Let

$$A_p := \sqrt{2}\left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\right)^{1/p}, \tag{2.1}$$

for $p > p_0 \approx 1.847$ and

$$A_p := 2^{\frac{1}{2}-\frac{1}{p}} \tag{2.2}$$

for $p \leq p_0 \approx 1.847$. The exact definition of p_0 is given by the following equality: $p_0 \in (1,2)$ is the unique real number with

$$\Gamma\left(\frac{p_0+1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

The constants A_p are the best constants satisfying Khinchine's inequality (due to U. Haagerup [6]). Up to now, the best constants satisfying the multilinear Bohnenblust–Hille inequality for real scalars appeared in [13] and obey the following recursive formula:

$$C_m = \begin{cases} 1 & \text{if } m = 1, \\ \left(A^{m/2}_{\frac{2m}{m+2}}\right)^{-1} C_{\frac{m}{2}} & \text{if } m \text{ is even, and} \\ \left(A^{\frac{-1-m}{2}}_{\frac{2m-2}{m+1}} C_{\frac{m-1}{2}}\right)^{\frac{m-1}{2m}} \left(A^{\frac{1-m}{2}}_{\frac{2m+2}{m+3}} C_{\frac{m+1}{2}}\right)^{\frac{m+1}{2m}} & \text{if } m \text{ is odd.} \end{cases} \tag{2.3}$$

For complex scalars the best known constants satisfying the multilinear Bohnenblust–Hille inequality appear in [10], and given by the formula

$$\widetilde{C}_m = \begin{cases} 1 & \text{if } m = 1, \\ \left(\left(\widetilde{A_{\frac{2m}{m+2}}}\right)^{m/2}\right)^{-1} \widetilde{C}_{\frac{m}{2}} & \text{if } m \text{ is even, and} \\ \left(\left(\widetilde{A_{\frac{2m-2}{m+1}}}\right)^{\frac{-1-m}{2}} \widetilde{C}_{\frac{m-1}{2}}\right)^{\frac{m-1}{2m}} \left(\left(\widetilde{A_{\frac{2m+2}{m+3}}}\right)^{\frac{1-m}{2}} \widetilde{C}_{\frac{m+1}{2}}\right)^{\frac{m+1}{2m}} & \text{if } m \text{ is odd,} \end{cases} \tag{2.4}$$

where

$$\widetilde{A_p} = \left(\Gamma\left(\frac{p+2}{2}\right)\right)^{\frac{1}{p}}.$$

3. New upper estimates: the exhaustive combinatorial approach

3.1. **Real case.** Let $f : [1,2)^2 \to \mathbb{R}$ be given by

$$f(x,y) = \frac{4x - 2xy}{4x + 4y - 4xy},$$

$r : \mathbb{N} \to \mathbb{R}$ be defined by

$$r(x) = \frac{2x}{1+x}$$

and $A : [1,2) \to \mathbb{R}$ be given by

$$A(p) = \begin{cases} 2^{\frac{1}{2}-\frac{1}{p}}; \text{ if } p \le p_0 \\ \sqrt{2}\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}\right)^{\frac{1}{p}}; \text{ if } p > p_0. \end{cases}$$

From [3, Theorem 4.1] and using the best known constants for the Khinchine inequality from [6] we can see that the optimal constants $(K_m)_{m=1}^{\infty}$ satisfying the real multilinear Bohnenblust–Hille inequality are such that

$$K_m \le J(k,m),$$

for all $k = 1, ..., \frac{m}{2}$ (when m is even) and $k = 1, ..., \frac{m-1}{2}$ (when m is odd), with

$$J(k,m) := \left(K_k \times (A(r(k)))^{k-m}\right)^{f(r(k),r(m-k))} \times \left(K_{m-k} \times (A(r(m-k)))^{-k}\right)^{f(r(m-k),r(k))}.$$

So, formally, the best estimate furnished by this method is

$$\begin{cases} K_m \le P_m := \min\left\{J(k,m) : k = 1, ..., \frac{m}{2}\right\} \text{ if } m \text{ is even} \\ K_m \le P_m := \min\left\{J(k,m) : k = 1, ..., \frac{m-1}{2}\right\} \text{ if } m \text{ is odd.} \end{cases} \tag{3.1}$$

A first inspection shows that the choice

$$\begin{cases} k = \frac{m}{2} \text{ for } m \text{ even,} \\ k = \frac{m-1}{2} \text{ for } m \text{ odd} \end{cases} \tag{3.2}$$

seems to be the best possible (i.e., the choice where the minimum of $J(k,m)$ is achieved). For this reason, in [13] this approach was selected and the formula (2.3) was presented. In [4, 11, 12] this formula was investigated in detail and now we know that

$$C_m \le \sqrt{2}\,(m-1)^{0.526322}.$$

As we mentioned before, using some numerical tests it seems clear that, in general, the choice (3.2) is better than other choices for k; for instance, the choice $k = 2$ was investigated in [9]. However, in some isolated cases we now identified that this choice of k given by (3.2) was not the best one. For this reason, in this paper we look for the sharper constants by using the whole formula (3.1). In view of the amount of calculations involved and since a reasonable precision in the decimals is crucial, this new approach was done with a computer program. The program, which code is in the Appendix, calculates the constants by using the formula (3.1). The first improvement on the constants appear for $m = 26$ and since it is a recursive procedure, this improvement generates improvements in several other values of m. The following table is illustrative:

m	new constants P_m	C_m (from [13])
26	< 5.22772	> 5.22825
27	< 5.31314	> 5.31447
28	< 5.39343	> 5.39626
29	< 5.47164	> 5.47314
100	< 10.509	> 10.510

From $m = 27$ to 500 the only values of m for which the new constants P_m are not strictly smaller than C_m are 31, 32, 33, 47, 48, 49, 63, 64, 65, 95, 96, 97, 127, 128, 129, 191, 192, 193, 255, 256, 257, 383, 384, 385. As m goes to infinity, it is natural that the exhaustive combinatorial approach achieves likely more constants. Moreover, for certain higher values of m, the difference $C_m - P_m$ can be chosen arbitrarily large, as the following result illustrates:

Proposition 3.1. *Given any increasing sequence of positive real numbers $(L_j)_{j=1}^{\infty}$ with $\lim_{j\to\infty} L_j = \infty$, there is a strictly increasing sequence $(m_k)_{k=1}^{\infty}$ of positive integers so that*

$$C_{m_j} - P_{m_j} > L_j \text{ for all } j.$$

Proof. Since the sequence $\left(A_{\frac{2m}{m+2}}^{-m/2}\right)_{m=1}^{\infty}$ is increasing (see [11, Lemma 6.1]) and since

$$\frac{2m}{m+1} > 1.85 > 1.847$$

for $m = 26$, we have

$$A_{\frac{2m}{m+2}}^{-m/2} = \left(\sqrt{2}\left(\frac{\Gamma\left(\frac{\frac{2m}{m+2}+1}{2}\right)}{\sqrt{\pi}}\right)^{(m+2)/2m}\right)^{\frac{-m}{2}}$$

when $m = 26$, and a direct calculation shows us that

$$1.415 < A_{\frac{52}{28}}^{-26/2} < 1.416.$$

Let $\varepsilon > 0$ be such that

$$C_{26} - P_{26} > \varepsilon.$$

Note that we can choose

$$\varepsilon = 10^{-4}.$$

From the recursive formulation of $(C_m)_{m=1}^{\infty}$ and from the fact $\left(A_{\frac{2m}{m+2}}^{-m/2}\right)_{m=1}^{\infty}$ is increasing we also have

$$C_{26\times 2^n} = d_1 \dots d_n C_{26}$$

and

$$P_{26\times 2^n} \leq d_1 \dots d_n P_{26},$$

with

$$1.415 < A_{\frac{52}{28}}^{-26/2} \leq d_1 \leq d_2 \leq \cdots$$

We thus have

$$\begin{aligned} C_{26\times 2^n} - P_{26\times 2^n} &\geq d_1 \dots d_n C_{26} - d_1 \dots d_n P_{26} \\ &> 10^{-4} \cdot (1.415)^n. \end{aligned}$$

Since $\lim_{n\to\infty} 10^{-4} \cdot (1.415)^n = \infty$, the proof is done. □

The following table illustrates the previous result:

m	$C_m - P_m$
$26 \cdot 2^{50}$	> 3450
$26 \cdot 2^{100}$	$> 1.19 \cdot 10^{11}$
$26 \cdot 2^{150}$	$> 4.10 \cdot 10^{18}$

3.2. **Complex case.** For complex scalars we made a similar exhaustive combinatorial approach up to $m = 500$ but the constants obtained were exactly the previous from [10], where the choice (3.2) is made. So, it seems that no improvement can be obtained following this argument and the question on the optimality of these constants is still open.

4. Appendix: the codes

In the code below, for real scalars, note that we replaced p_0 by 1.846999. We remark that this procedure does not cause any problem (no lack of precision in the estimates). The reason is simple. In fact, since the function r is increasing and

$$r(12) = \frac{24}{13} < 1.8463 < p_0$$
$$r(13) = \frac{26}{14} > 1.857 > p_0,$$

there is absolutely no difference in working with 1.846999 instead of p_0. As a matter of fact, we could have even used 1.847 instead of 1.846999. The new constants, up to 500, can be easily checked by means of, for instance, the code given below (that was made using the *Mathematica* package and provides the first 500 values of the constants).

- Code for the Real case:

```
M:=500;digits=100;
f[x_,y_]:=(4*x-2*x*y)/(4*x+4*y-4*x*y);
r[x_]:=(2*x)/(1 + x);
A[k_]:=
  If[k>1.846999,
  Sqrt[2]*(Gamma[(k+1)/2]/Sqrt[Pi])^(1/k),
  2^(1/2-1/k)];
CBH[1]:=1;
CBH[2]:=Sqrt[2];
For[m=3,m<=M,m++,
 CBH[m]=
  Min[
   Table[
    N[((CBH[k]*A[r[k]]^(k-m))^(f[r[k],r[m-k]]))*
    ((CBH[m-k]*A[r[m-k]]^(-k))^(f[r[m-k],r[k]])),digits],
    {k,1,If[Mod[m,2]==0,m/2,(m-1)/2]}]
   ];
 Print[{m,CBH[m]}];
]
```

- Code for the Complex case:

```
M:=500;digits=100;
f[x_,y_]:=(4*x-2*x*y)/(4*x+4*y-4*x*y);
r[x_]:=(2*x)/(1+x);
A[k_]:=Gamma[(k+2)/2]^(1/k);
CBH[1]:=1;
CBH[2]:=2/Sqrt[Pi];
For[m=3,m<=M,m++,
 CBH[m]=
  Min[
  Table[
    N[((CBH[k]*A[r[k]]^(k - m))^(f[r[k],r[m-k]]))*
    ((CBH[m-k]*A[r[m-k]]^(-k))^(f[r[m-k],r[k]])),digits],
    {k, 1,If[Mod[m,2]==0,m/2,(m-1)/2]}]
```

```
    ];
   Print[{m,CBH[m]}];
  ]
```

References

[1] H.F. Bohnenblust and E. Hille, On the absolute convergence of Dirichlet series, Ann. of Math. (2) **32** (1931), 600–622.
[2] A.M. Davie, Quotient algebras of uniform algebras, J. London Math. Soc. **7** (1973), 31–40.
[3] A. Defant, D. Popa and U. Schwarting, Coordinatewise multiple summing operators in Banach spaces, J. Funct. Anal. **259** (2010), 220–242.
[4] D. Diniz, G.A. Muñoz-Fernández, D. Pellegrino and J.B. Seoane-Sepúlveda, The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal, J. Funct. Anal. **263** (2012), 415–428.
[5] D. Diniz, G.A. Muñoz-Fernández, D. Pellegrino and J.B. Seoane-Sepúlveda, Lower bounds for the constants in the Bohnenblust–Hille inequality: the case of real scalars, Proc. Amer. Math. Soc., accepted for publication.
[6] U. Haagerup, The best constants in the Khinchin inequality, Studia Math. **70** (1982), 231–283.
[7] S. Kaijser, Some results in the metric theory of tensor products, Studia Math. **63** (1978), 157–170.
[8] A. Montanaro, Some applications of hypercontractive inequalities in quantum information theory, J. Math. Phys. 53, 122206, 2013.
[9] G.A. Muñoz-Fernández, D. Pellegrino and J.B. Seoane-Sepúlveda, Estimates for the asymptotic behaviour of the constants in the Bohnenblust Hille inequality, Linear Multilinear Algebra **60** (2012), 573–582.
[10] D. Nuñez-Alarcón, D. Pellegrino, J.B. Seoane-Sepúlveda, On the Bohnenblust–Hille inequality and a variant of Littlewood's 4/3 inequality, J. Funct. Anal. **264** (2013), 326–336.
[11] D. Nuñez-Alarcón, D. Pellegrino, J.B. Seoane-Sepúlveda, D.M. Serrano-Rodríguez, There exist multilinear Bohnenblust–Hille constants $(C_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty}(C_{n+1}-C_n)=0$, J. Funct. Anal. **264** (2013), 429–463.
[12] D. Nuñez-Alarcón, D. Pellegrino, J.B. Seoane-Sepúlveda, D.M. Serrano-Rodríguez, On the optimal multilinear Bohnenblust–Hille constants, arXiv:1302.0517.
[13] D. Pellegrino and J.B. Seoane-Sepúlveda, New upper bounds for the constants in the Bohnenblust Hille inequality, J. Math. Anal. Appl. **386** (2012), 300–307.
[14] H. Queffélec, H. Bohr's vision of ordinary Dirichlet series: old and new results, J. Anal. **3** (1995), 43–60.
[15] D.M. Serrano-Rodríguez, Improving the closed formula for subpolynomial constants in the multilinear Bohnenblust–Hille inequalities, Linear Alg. Appl. **438** (2013), 3124–3138.

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